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On the Lagrangian and Hamiltonian constraint algorithms for the Rarita–Schwinger field coupled to an external electromagnetic field

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Abstract. The complete constraint analysis for the Rarita-Schwinger $\frac{3}{2}$ field coupled to an external electromagnetic field is given, showing the parallel between the Lagrangian and Dirac-Bergman algorithms. The analysis confirms the connection between the Velo-Zwanzinger and Johnson-Sudarshan pathologies, and illustrates that these do not, in fact, arise because the constraint analysis leading to them is incomplete. On the contrary, a new tier of constraints occurs for the critical field values, reducing the pathology to a field-induced change in degrees of freedom.

1. Introduction

Some time ago Hasumi *et al* (1979, hereafter referred to as HEK) studied the Dirac quantisation of a massive spin-$\frac{3}{2}$ particle coupled to an external magnetic field. Apart from the intrinsic interest of the Dirac constraint quantisation applied to such a coupled field, HEK were also interested in the manifestation of the Johnson-Sudarshan inconsistency in the Dirac algorithm (Johnson and Sudarshan 1961, hereafter referred to as JS). Johnson and Sudarshan pointed out that that anticommutators for the Rarita-Schwinger field coupled to an external electromagnetic field are indefinite. This well known inconsistency has been throughly discussed but HEK examined its origins carefully, finding that the constraint analysis by which it is normally obtained was not complete. For certain values of the external field the secondary constraint used to obtain the 'true equation of motion' leading to the JS problem degenerates and a further heirarchy of constraints appears, which has to be analysed. This results in a loss of two degrees of freedom and the final anticommutator in such cases is much more complicated than that of JS.

Another well known Rarita-Schwinger coupling inconsistency is that of Velo and Zwanzinger (1969, hereafter referred to as VZ), which occurs even at the classical level. VZ showed that, for certain values of the external field, the field equations either propagate acausal modes or even do not propagate at all. The VZ analysis is done covariantly by a Lagrangian constraint algorithm, but the problems again surface when the secondary constraints are used to get the true equation of motion. At about the same time as the HEK analysis Takahashi and Kobayashi (1978), when discussing the connections between the JS and VZ problems and the Gribov ambiguity, also noted the degeneracy of the secondary constraint at a critical field value, referring to the

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same observation in earlier work by Jenkins (1974). The values of the field at the onset of the VZ and JS problems coincide, indicating an underlying common origin for both. Indeed Kobayashi and Takahashi (1987) have recently explored this question by a modified Dirac constraint algorithm, and have identified the common source of the difficulty as the invertibility condition required for a unique solution of the secondary constraint. However, these authors did not proceed past the invertibility condition in the case when it is violated, which is precisely the case for the values of the field yielding acausality in the VZ analysis. Effectively, they stopped short of the full constraint analysis employed by HEK in pursuit of the JS problem, in which the next hierarchy of the constraint algorithm is entered for the critical values of the external field.

In this paper we complete the VZ Lagrangian constraint analysis, in direct analogy to HEK. There is again a loss of degrees of freedom, pre-empting the VZ causality problem. This of course offers no resolution of the paradoxes to which the coupled Rarita–Schwinger field is prone, but it gives an understanding of their deeper structure. Since the HEK analysis is via the Dirac–Bergmann constraint algorithm (as is that of Kobayashi and Takahashi), while the VZ treatment is via a Lagrangian constraint analysis it is necessary to exhibit the correspondence between these two constraint algorithms. The equivalence between the Lagrangian and Hamiltonian constraint algorithms for Lagrangians yielding second-order equations of motion has been shown, in general, in the work of Gotay and Nester (1979) and Batlle *et al* (1986). The modification of this equivalence theorem for first-order Lagrangians such as the Rarita–Schwinger example has been presented by Scherer (1986) in coordinate-dependent form and by Cariñena *et al* (1988) in geometric form. The analysis given here provides an interesting example of this equivalence.

2. The Lagrangian and Hamiltonian constraint analysis for first-order Lagrangians

The general features of the Lagrangian and Hamiltonian constraint analysis procedures are well known and are described, for finite-dimensional dynamical systems, in Sudarshan and Mukunda (1974), for example. The case in which the Lagrangian is linear in the velocities has been treated in detail by Scherer (1986), who shows the precise connection between the Lagrangian and Hamiltonian constraints for such Lagrangians. The extension to field theories is formally straightforward (although the occurrence of spatial derivatives in the constraints demands careful consideration of spatial boundary values in a rigorous treatment (Sundermeyer 1982)) and is briefly summarised here for convenience.

We consider a Lagrangian with the spacetime decomposed form

$$\mathscr{L} = \phi^r A^0_{rs}(\phi)\partial_0\phi^s - H(\phi, \partial_k\phi) \tag{2.1}$$

for component fields ϕ^r, $r = 1, \ldots, n$.

The Euler–Lagrange equations are

$$B^{(0)}_{rs}(\phi)\partial_0\phi^s = b^{(0)}_r(\phi) \tag{2.2}$$

where

$$B^{(0)}_{rs}(\phi) = A^0_{rs} - A^0_{sr} + \phi^t \frac{\partial A^0_{ts}}{\partial \phi^r} - \phi^t \frac{\partial A^0_{tr}}{\partial \phi^s} \tag{2.3}$$

and

$$b_r^{(0)}(\phi)=\frac{\partial H}{\partial \phi^r}-\partial_k\left(\frac{\partial H}{\partial(\partial_k\phi^r)}\right). \tag{2.4}$$

For a singular system the determinant of the coefficients in (2.2) is zero:

$$|B_{ij}^{(0)}|=0. \tag{2.5}$$

Not all the 'velocities' $\partial_0\phi^s$ can be determined, and constraints arise.

The Lagrangian constraint algorithm consists of determining the left null eigenspace of $B^{(0)}$ and evaluating the equation (2.2) on this eigenspace, thereby obtaining a set of Lagrangian primary constraints

$$\chi_\alpha(\phi)=0 \qquad \alpha=1,2,\ldots,K^{(0)}. \tag{2.6}$$

(In general this stage may be reached only after an iterative process in which constraints achieved at one step may modify the null eigenspace of $B^{(0)}$, when it is determined on the corresponding constraint submanifold, resulting in the production of new constraints when consistency with (2.2) is required.)

The next step in the Lagrangian analysis is to demand that the constraints (2.6) be preserved in time on the constraint hypersurface. This produces new equations for the velocities, $\partial_0\phi^s$, which must be reconsidered in conjunction with (2.2) as a new system of equations

$$B_{sj}^{(1)}\partial_0\phi^j=b_s^{(1)} \qquad s=1,\ldots,n+K^{(0)} \tag{2.7}$$

with

$$B^{(1)}=\begin{bmatrix} B_{ij}^{(0)} \\ \partial\chi_\alpha/\partial\phi^j\end{bmatrix} \qquad b^{(1)}=\begin{bmatrix} b_i^{(0)} \\ 0 \\ \vdots \\ 0\end{bmatrix}. \tag{2.8}$$

By examining the left null eigenspace of $B_{sj}^{(1)}$ and demanding consistency of (2.7) on this eigenspace we may determine more of the velocities and may also obtain further secondary constraints. Repeating this procedure as necessary the final situation reached can be arranged in the form of K Lagrangian constraints:

$$\Gamma_\alpha(\phi)=0 \qquad \alpha=1,\ldots,K \tag{2.9}$$

and $n+K$ equations for the velocities

$$B_{sj}\partial_0\phi^j=b_s \qquad s=1,\ldots,(n+K) \tag{2.10}$$

where

$$B=\begin{bmatrix} B^{(0)} \\ \partial\Gamma_\alpha/\partial\phi^j\end{bmatrix} \qquad b=\begin{bmatrix} b^{(0)} \\ 0 \\ \vdots \\ 0\end{bmatrix} \tag{2.11}$$

where $B^{(0)}$, $b^{(0)}$ are as defined in the Euler–Lagrange equations (2.2). The left null eigenvectors of B produce no new constraints which are independent of the Γ_α. If, finally, rank $B=R<n$ then R of the velocities $\partial_0\phi^j$ can be uniquely determined while the remaining $n-R$ appear as arbitrary functions in the solutions to the equations of motion.

In the corresponding Dirac-Bergmann Hamiltonian constraint analysis of (2.1) the definition of the canonical momenta fields as

$$\pi_r = \frac{\partial \mathscr{L}}{\partial(\partial_0 \phi^r)} \tag{2.12}$$

yields, for the singular case, the Hamiltonian primary constraints

$$\gamma_r = \pi_r - \phi^s A^0_{sr}(\phi) = 0. \tag{2.13}$$

Introducing Lagrange multipliers λ^r, a primary constraint Hamiltonian density is now defined as

$$\mathscr{H}_c(\phi, \pi) = H(\phi, \partial_k \phi) + \lambda^r \gamma_r \tag{2.14}$$

and is used to guarantee time preservation of the primary constraints

$$\gamma_r = \{\gamma_r, \mathscr{H}_c\} \approx 0 \tag{2.15}$$

$\{\, ,\}$ denoting the functional Poisson bracket and $\approx$ denoting weak equality. This results in the equations

$$B^{(0)}_{ij}(\phi)\lambda^j = b^{(0)}_i(\phi) \tag{2.16}$$

$B^{(0)}_{ij}$, $b^{(0)}_i$ being as defined in (2.3) and (2.4). The equality in (2.16) may be taken as strong since the constraints (2.13) restrict only the momenta fields, π_r, which do not occur in (2.16). Equation (2.16) is identical to the Euler-Lagrange equation (2.2), but the Lagrange multipliers λ^i now play the role of the velocities.

The constraint structure of (2.16) may now be analysed in direct analogy to the Lagrangian constraint analysis of (2.2). The condition that (2.16) has solutions for the λ^i produces a set of secondary Hamiltonian constraints

$$\chi_\alpha(\phi) = 0 \qquad \alpha = 1, \ldots, K^{(0)} \tag{2.17}$$

identical to the Lagrangian primary constraints (2.6). Time preservation of (2.17) and repeated iteration of the constraint analysis now follows analogous steps to the Lagrangian case, with time preservation now ensured via

$$\{\text{constraint}, \mathscr{H}_c\} = 0.$$

The problem is now one of determing the λ^i rather than the $\partial_0 \phi^i$. The final situation in the Dirac-Bergmann algorithm is as follows.

There are n Hamiltonian primary constraints

$$\gamma_i(\phi, \pi) = 0$$

K r-ary ($r \geqslant 2$) Hamiltonian constraints

$$\Gamma_\alpha(\phi) = 0 \qquad \alpha = 1, \ldots, K$$

and the multipliers obey the equations

$$B_{sj}\lambda^i = b_s$$

with B and b as defined in the Lagrangian case (2.11).

The conditions for existence of unique solutions for the λ^i yield no new constraints independent of the Γ_α. The $(r+1)$-ary Hamiltonian constraints of the Dirac-Bergmann algorithm are the same as the r-ary constraints of the Lagrangian analysis.

3. Application to the Rarita–Schwinger equation coupled to an external electromagnetic field

The Lagrangian density for the spin-$\frac{3}{2}$ Rarita–Schwinger field ψ_μ coupled to an external electromagnetic field A_μ is

$$\mathscr{L}=\tfrac{1}{2}\mathrm{i}\varepsilon^{\lambda\rho\mu\nu}\bar{\psi}_\lambda\gamma^5\gamma_\mu\partial_\nu\psi_\rho-\tfrac{1}{2}\mathrm{i}\varepsilon^{\lambda\rho\mu\nu}\partial_\nu\bar{\psi}_\lambda\gamma^5\gamma_\mu\psi_\rho+m\bar{\psi}_\lambda\sigma^{\lambda\rho}\psi_\rho+e\varepsilon^{\lambda\rho\mu\nu}\bar{\psi}_\lambda\gamma^5\gamma_\mu\psi_\rho A_\nu \tag{3.1}$$

where

$$\gamma^5=\gamma^0\gamma^1\gamma^2\gamma^3 \qquad \bar{\psi}_\lambda=\psi^\dagger_\lambda\gamma^0 \qquad \sigma^{\mu\nu}=\tfrac{1}{2}(\gamma^\mu\gamma^\nu-\gamma^\nu\gamma^\mu)$$

and the conventions are

$$g^{\mu\nu}=(1,-1,-1,-1)=g_{\mu\nu} \qquad \varepsilon^{0123}=1=-\varepsilon_{0123}.$$

The Euler–Lagrange field equation resulting from varying $\bar{\psi}_\mu$ in (3.1) is perhaps best written in the form

$$E_\kappa=\gamma\cdot\pi\psi_\kappa-\gamma_\kappa\pi\cdot\psi-\pi_\kappa\gamma\cdot\psi+\gamma_\kappa\gamma\cdot\pi\gamma\cdot\psi-m\psi_\kappa+m\gamma_\kappa\gamma\cdot\psi=0 \tag{3.2}$$

where

$$a\cdot b\equiv a^\mu b_\mu$$

and

$$\pi_\mu=\mathrm{i}\partial_\mu+eA_\mu.$$

As is well known, a spacetime decomposition of (3.2) reveals the absence of the 'velocity' $\partial_0\psi_0$. In particular the $\kappa=0$ component of (3.2) gives a Lagrangian primary constraint

$$\boldsymbol{\pi}\cdot\boldsymbol{\psi}-\boldsymbol{\gamma}\cdot\boldsymbol{\pi}\boldsymbol{\gamma}\cdot\boldsymbol{\psi}-m\boldsymbol{\gamma}\cdot\boldsymbol{\psi}=0 \tag{3.3}$$

where $\boldsymbol{a}=(a_1,a_2,a_3)$ and $\boldsymbol{a}\cdot\boldsymbol{b}=a^ib_i$, $i=1,2,3$. Equation (3.3) contains no field time derivatives at all. The spatial components of (3.2), $\kappa=1, 2, 3$ yield equations for the time derivatives $\partial_0\psi_i$.

Before proceeding to the next stage in the constraint analysis we must check that (3.3) is the only primary constraint resulting from algebraic operations on the field equations (3.2). The only such operations open to us are multiplications and contractions with $g^{\mu\nu}$ and the independent elements of the Dirac algebra, γ_μ, $[\gamma_\mu,\gamma_0]$, $\gamma_5\gamma_\mu$, γ_5. It is easily verified that all such operations on E_κ amount to either (3.3) or contraction with γ^κ, which produces

$$2(\gamma\cdot\pi\gamma-\pi)\cdot\psi+3m\gamma\cdot\psi=0 \tag{3.4}$$

which is not a constraint but simply a useful relation between the velocities $\partial_0\psi_i$. Thus (3.3) is the only primary constraint, corresponding to (2.6).

One must now examine the conditions arising from time preservation of (3.3). Since (3.3) is, in fact, already implicit in (3.2) it is convenient to maintain manifest covariance and contract (3.2) with π_μ, which results in

$$m((\gamma\cdot\pi)\gamma-\pi)\cdot\psi-\frac{\mathrm{i}e}{2}\gamma^5\varepsilon_{\kappa\nu\mu\lambda}\gamma^\nu F^{\mu\kappa}\psi^\lambda=0 \tag{3.5}$$

where $F^{\mu\nu}=\partial^\mu A^\nu-\partial^\nu A^\mu$.

Equation (3.5) contains field time derivatives, and when combined with the original equations (3.2), corresponds to equation (2.7), from which we must extract the essential secondary constraints. Algebraic manipulation of (3.2) and (3.5), via (3.4), thus results in the well known secondary constraint

$$\gamma\cdot\psi+\lambda\gamma^5\varepsilon_{\kappa\nu\mu\lambda}\gamma^\nu F^{\mu\kappa}\psi^\lambda=0 \tag{3.6}$$

where $\lambda=\mathrm{i}e/3m^2$. Equation (3.6) guarantees the time preservation of the primary constraint (3.3).

Now the constraint analysis will be complete if (3.6), in conjunction with the original field equation (3.2), allows determination of the time derivative of ψ_0. This was the problem investigated by Velo and Zwanzinger, by substituting (3.6) back into the field equations to obtain a 'true equation of motion':

$$(\gamma\cdot\pi-m)\psi_\kappa+(\pi_\kappa+\tfrac{1}{2}m\gamma_\kappa)\lambda\gamma^5\varepsilon_{\kappa\nu\mu\lambda}\gamma^\nu F^{\mu\kappa}\psi^\lambda=0. \tag{3.7}$$

Written as a first-order matrix differential equation this becomes

$$[(M^\mu)^\lambda_\kappa\partial_\mu-B^\lambda_\kappa]\psi_\lambda=0 \tag{3.8}$$

where the derivative coefficients are

$$[M^\mu]^\lambda_\kappa=g_\kappa{}^\lambda\gamma^\mu+\lambda\gamma^5\varepsilon_{\alpha\nu\rho}{}^\lambda\gamma^\nu F^{\rho\alpha}g_\kappa{}^\mu. \tag{3.9}$$

The characteristic surfaces for the system (3.8) have normals n_μ determined by

$$D(n)=|[M^\mu]^\lambda_\kappa n_\mu|=0. \tag{3.10}$$

The well known acausality problems of Velo–Zwanzinger arise if there are spacelike characteristic surfaces—that is, if there exist timelike normals n_μ. By Lorentz covariance we can seek such normals in the form $(n, 0, 0, 0)$ in the forward light cone, and then (3.10) simplifies to

$$D(n)=n^{16}|M^0|=0 \tag{3.11}$$

where

$$[M^0]^\lambda_\kappa=g_\kappa{}^\lambda\gamma^0+\lambda\gamma^5\varepsilon_{\alpha\nu\rho}{}^\lambda\gamma^\nu F^{\rho\alpha}g_\kappa{}^0. \tag{3.12}$$

Thus, if there are values of the external field $F^{\mu\nu}$ such that $|M^0|=0$, then (3.11) allows non-zero solutions for n, timelike normals and therefore spacelike characteristics exist, and acausality ensues. The condition $|M^0|=0$ reduces to

$$|\gamma^0|^3|\gamma^0+\lambda\gamma^5\varepsilon_{\alpha\nu\rho}{}^0\gamma^\nu F^{\rho\alpha}|=0=(1+4\lambda^2\boldsymbol{B}^2)^2=\left[1-\left(\frac{2e}{3m^2}\right)^2\boldsymbol{B}^2\right]^2=0 \tag{3.13}$$

and, as shown by Velo and Zwanzinger, this allows values of the external field leading to acausal propagation.

However, the above analysis of vz is incomplete. For precisely the case which allows the occurrence of spacelike characteristic surfaces, namely $|M^0|=0$, ensures that the constraint (3.6) fails to determine all the components of ψ^0, and so (3.6) is then not the end of the constraint analysis. Thus, the coefficient of ψ^0 in (3.6) is the factor

$$\gamma^0+\lambda\gamma^5\varepsilon_{\alpha\nu\rho}{}^0\gamma^\nu F^{\rho\alpha} \tag{3.14}$$

which occurs in (3.13), and if this operator is singular, as required by (3.13) then (3.6) fails to determine all the components of ψ_0. This point was observed by Takahashi and Kobayashi (1978) and Jenkins (1974), but taken no further. The vz prediction of acausality is therefore premature because, under those circumstances for which it occurs, the constraint analysis is still not complete. The fact that (3.6) is not the last word in the constraint analysis was noted by HEK in the context of the Dirac-Bergmann constraint analysis of the JS problem, although they made no reference to the VZ problem. Since we now know that these two pathologies are related (Kobayashi and Takahashi 1987) in the Dirac-Bergmann analysis, then the HEK treatment of the JS problem must have a parallel in the VZ Lagrangian analysis. Incidentally, the factor (3.14) corresponds essentially to the operator $(dc^{\dagger})$ of Kobayashi and Takahashi, which is identified as the common source of the JS and VZ problems, being the subject of the invertibility condition which generates these problems. However, like VZ and JS they do not proceed past the secondary constraint (3.6), leaving the analysis incomplete. Following in parallel to the HEK analysis of the JS problem we now complete the Lagrangian analysis by looking at the time preservation of (3.6) for the case when the factor (3.14) is singular.

Following HEK, we assume for simplicity that $F_{io} = 0$ and $F_{ij} =$ constant, and write (3.6) in the form

$$2R\gamma_0\psi_0 + \Gamma^\kappa\psi_\kappa = 0 \tag{3.15}$$

where

$$R = \tfrac{1}{2}[1 - \lambda\sigma^{ij}F_{ij}] \tag{3.16}$$

$$\Gamma^\kappa = \gamma^\kappa + \lambda\gamma^\lambda\gamma^\kappa\gamma^\rho F_{\lambda\rho} \tag{3.17}$$

and $2R\gamma_0$ is the operator (3.14). If R is non-singular then (3.15) determines all the ψ_0 in terms of the ψ_i, and since all the time derivatives of the latter are determined by (3.2), all 'velocities' are determined and the constraint analysis is complete. The operator R satisfies the equation

$$R^2 - R + \tfrac{1}{4}(1 + 4\lambda^2\boldsymbol{B}^2) = 0 \tag{3.18}$$

where $\boldsymbol{B}$ is the magnetic field $B^i = \tfrac{1}{2}\varepsilon^{ijk}F_{jk}$, and

$$|R| = \left(\frac{1 + 4\lambda^2\boldsymbol{B}^2}{4}\right)^2. \tag{3.19}$$

Thus, if R is singular then

$$1 + 4\lambda^2\boldsymbol{B} = 0 \tag{3.20}$$

and R becomes a projection operator:

$$R^2 = R. \tag{3.21}$$

Equation (3.20) is the putative VZ condition for acausal propagation, as noted above.

Defining

$$\tilde{R} = I - R = \tfrac{1}{2}(1 + \lambda\sigma^{ij}F_{ij}) \tag{3.22}$$

we project the secondary constraint (3.15) into

$$2R\gamma_0\psi_0 + R\Gamma^k\psi_k = 0 \tag{3.23}$$

$$\tilde{R}\Gamma^k\psi_k = 0. \tag{3.24}$$

Thus, when R is singular, only the projection $R\psi_0$ is determined by the secondary constraint. To determine $\tilde{R}\psi_0$ we must consider the time preservation of (3.24)—the time preservation of (3.23) clearly tells us nothing about $\tilde{R}\psi_0$:

$$\tilde{R}\Gamma^k\dot{\psi}_k = 0. \tag{3.25}$$

This new equation for the 'velocities' must now be considered in conjunction with the other such equations, namely the original field equations (3.2), the $\kappa = k$ component of which gives, on application of the constraint (3.23),

$$\mathrm{i}\gamma^0\dot{\psi}_k = (\pi_k + \tfrac{1}{2}m\gamma_k)\gamma_0\psi_0 - (\boldsymbol{\gamma}\cdot\boldsymbol{\pi} - m)\psi_k + (\pi_k + \tfrac{1}{2}m\gamma_k)\gamma^i\psi_i \tag{3.26}$$

where, without loss of generality, we have taken $A_0 = 0$ for convenience.

Substituting (3.26) into (3.25), and using the results

$$\Gamma^k = R\gamma^k + \gamma^k R \tag{3.27}$$

$$\Gamma^k(\pi_k + \tfrac{1}{2}m\gamma_k) = \boldsymbol{\gamma}\cdot\boldsymbol{\pi}R + R\boldsymbol{\gamma}\cdot\boldsymbol{\pi} + m(1+R) \tag{3.28}$$

$$\tilde{R}R = 0 \tag{3.29}$$

and the constraint (3.23), finally results in the tertiary constraint

$$\tilde{R}\gamma_0\psi_0 + \tilde{R}\Lambda^k\psi_k = 0 \tag{3.30}$$

where

$$\Lambda^k = \gamma^k + \frac{2e^2}{m(3m^2)^2}F_{nm}\pi^n\gamma^m F_{ij}\gamma^i g^{jk} - \frac{4\mathrm{i}e}{3m^2}F_{ij}\pi^i g^{jk}. \tag{3.31}$$

Equation (3.30) determines $\tilde{R}\psi_0$ and so, along with (3.23), ψ_0 is determined entirely in terms of ψ_k, the time derivatives of which are determined by (3.26), on substitution for ψ_0. However the new relation (3.24) between the ψ_k signals a loss of degrees of freedom due to the external field taking certain values—there are now six degrees of freedom, compared with the usual eight for a massive spin-$\frac{3}{2}$ field, a point noted by HEK.

To exemplify the Scherer connection between Lagrangian and Hamiltonian constraint analysis we briefly outline the HEK Dirac–Bergmann analysis.

The conjugate momenta to ψ^μ_a (a denotes the bispinor label) define primary Hamiltonian constraints:

$$\varphi^0_a \equiv \pi^0_a = 0 \tag{3.32}$$

$$\varphi^{0\dagger}_a \equiv \pi^{0\dagger}_a = 0 \tag{3.33}$$

$$\varphi^k_a \equiv \pi^k_a - \tfrac{1}{2}\mathrm{i}\varepsilon^{kij}(\psi^\dagger_i\gamma^5\gamma_j\gamma^0)_a = 0 \tag{3.34}$$

$$\varphi^{k\dagger}_a \equiv \pi^{k\dagger}_a - \tfrac{1}{2}\mathrm{i}\varepsilon^{kij}(\gamma^0\gamma^5\gamma_j\psi_i)_a = 0. \tag{3.35}$$

The primary constraint Hamiltonian density is

$$\mathscr{H}_c = H + \lambda_{\mu a}\varphi^\mu_a + \lambda^\dagger_{\mu a}\varphi^{\mu\dagger}_a \tag{3.36}$$

where H is the canonical Hamiltonian:

$$H = -\tfrac{1}{2}\mathrm{i}\varepsilon^{\lambda\rho\mu\nu}\bar{\psi}_\lambda\gamma^5\gamma_\mu\partial_\kappa\psi_0 + \tfrac{1}{2}\mathrm{i}\varepsilon^{\lambda\rho\mu\nu}\partial_\kappa\bar{\psi}_\lambda\gamma^5\gamma_\mu\psi_\rho - m\bar{\psi}_\lambda\sigma^{\lambda\rho}\psi_\rho - e\varepsilon^{\lambda\rho\mu\nu}\bar{\psi}_\lambda\gamma^5\gamma_\mu\psi_\rho A_\nu. \tag{3.37}$$

Now time preservation of the primary constraint (3.33) yields, on evaluation of $\{\varphi^{0\dagger}_a, \mathscr{H}_c\} = \{\pi^{0\dagger}_a, \mathscr{H}_c\} = 0$, the secondary Hamiltonian constraint

$$\sigma^{ij}\pi_i\psi_j + m\gamma^i\psi_i = 0 \tag{3.38}$$

which is identical to the primary Lagrangian constraint (3.3). Time conservation of (3.32) simply yields the conjugate to (3.38).

Time preservation of (3.35), on the other hand, produces the equation (again assuming $A_0 = 0$ without loss of generality)

$$\mathrm{i}\sigma^{lk}\lambda_l + \varepsilon^{okjl}\gamma^0\gamma^5\gamma_0\pi_l\psi_j + \sigma^{kl}\pi_l\psi_0 + m\gamma^0\sigma^{kp}\psi_p = 0$$

which, on use of the primary constraint (3.38), solves for the λ_k as

$$\mathrm{i}\gamma^0\lambda_k = -(\boldsymbol{\gamma}\cdot\boldsymbol{\pi} - m)\psi_k + (\pi_k + \tfrac{1}{2}m\gamma_k)\gamma^i\psi_i + (\pi_k + \tfrac{1}{2}m\gamma_k)\gamma^0\psi_0. \tag{3.39}$$

This confirms, with (3.26), that the Lagrange multipliers λ_k play the role of the field time derivative, $\lambda_k = \dot{\psi}_k$.

Now time preservation of the primary constraint (3.28) yields

$$\sigma^{ij}\pi_i\lambda_j + m\gamma^i\lambda_j = 0$$

whence substitution from (3.39) and use of (3.28) gives the tertiary Hamiltonian constraint

$$2\gamma^0 R\psi_0 + \Gamma^i\psi_i = 0$$

i.e. the secondary Lagrangian constraint (3.15), equivalent to (3.6).

The rest of the constraint analysis is now straightforward, parallel to the Lagrangian case. At each stage the $(r+1)$-ary constraints of the HEK analysis correspond to the r-ary constraints of the Lagrangian analysis, as in the general analysis of Scherer. The Lagrangian multipliers $\lambda_{\mu a}$ correspond to the field time derivatives and are determined in a similar way.

4. Conclusion

The full constraint analysis for the Rarita–Schwinger spin-$\frac{3}{2}$ field coupled to an external electromagnetic field has been given, showing the parallel between the Lagrangian and Dirac–Bergmann algorithms, as described for a general first-order system by Scherer (1986). It is shown that the VZ analysis, which found acausal propagation, is incomplete and that those values of the field which ostensibly lead to such problems in fact mark a degeneracy in the secondary constraint which is equivalent to a loss of degrees of freedom. This parallels a similar analysis of the Johnson and Sudarshan problem by Hasumi *et al* (1979), and effectively confirms the connection between the VZ and JS problems noted by Kobayashi and Takahashi (1987), but illustrates this right through the complete constraint analysis.

That the acausality and quantisation pathologies of VZ and JS, in fact, seem to degenerate to a loss of degrees of freedom problem is interesting. It means that the type of pathology which can occur may not be so wide as first thought, and it may be possible to concentrate attention on just the loss of degrees of freedom problem. Also, it would be interesting to know how general this feature is. No quantisable massive theory of a coupled high spin field is known to be problem free, but perhaps the problems are more restricted than previously believed.

One of the motivations for the present analysis is to set the VZ and JS analysis in the context of general Lagrangian and Dirac–Bergmann constraint algorithms and the connections between them for first-order singular systems. In recent years these constraint algorithms have been formulated in geometric terms (Cariñena *et al* 1988), effectively geometrising the Scherer analysis, and it should now be possible to translate the VZ/JS-type consistency analysis into a coordinate-independent geometrical form. Apart from possible insights this may give into high spin consistency analysis, it will also provide a connection with the predominantly geometric modern approach to massless gauge theories.

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